Field Theory on Infinitesimal-Lattice Spaces

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Abstract

Equivalence in physics is discussed on the basis of experimental data accompanied by experimental errors. It is pointed out that the introduction of the equivalence being consistent with the mathematical definition is possible only in theories constructed on non-standard number spaces by taking the experimental errors as infinitesimal numbers. Following the idea for the equivalence, a new description of space-time ${}^*\mathcal{L}$ in terms of infinitesimal-lattice points on non-standard real number space ${}^*\mathcal{R}$ is proposed. By using infinitesimal neighborhoos $(\mathrm{Mon}(r|\ {}^*\mathcal{L}))$ of real number r on ${}^*\mathcal{L}$ we can make a space ${}^*\mathcal{M}$ which is isomorphic to $\mathcal{R}$ as additive group. Therefore, every point on $({}^*\mathcal{M})^N$ automatically has the internal confined-subspace $\mathrm{Mon}(r|\ {}^*\mathcal{L})$. A field theory on ${}^*\mathcal{L}$ is proposed. It is shown that $U(1)$ and $SU(N)$ symmetries on the space $({}^*\mathcal{M})^N$ are induced from the internal substructure $(\mathrm{Mon}(r|\ {}^*\mathcal{L}))^N$. Quantized state describing configuration space is constructed on $({}^*\mathcal{M})^N$. We see that Lorentz and general relativistic transformations are also represented by operators which involve the $U(1)$ and $SU(N)$ internal symmetries.

1 Introduction

We would like to start from a question,

"Why are non-standard spaces needed in theories of physics?"

In our observations the judgment of equivalence between two or more phenomena plays a very important role. It is kown that the equivalence is rigorously defined in mathematics in terms of the following three conditions;
(1) $A \sim A$ (reflection)
(2) $A \sim B \Longrightarrow B \sim A$, (symmetry)
(3) $A \sim B, \quad B \sim C \Longrightarrow A \sim C$. (transitivity)
In observations of physics, that is, in experiments, the equivalence (physical equivalence) can be described as follows:

Two phenomena A *and* B *are equivalent,*
if A *and* B *coincide within the experimental errors.*

It should be stressed that the physical equivalence is detemined by the experimental errors. Futhermore we must recognize that there is no experiments accompanied by no error. We should consider that experimental errors are one of the fundamental observables in our

experiments. It is quite hard to understand that there is no theory which involves any description of experimental errors, even though they are very fundamental observables. It is also hard to understand that the question whether such physical equivalence is compatible with the mathematical definition represented by the above three conditions had never been discussed. Let us discuss the question here. We easily see that the first two conditions, that is, reflection and symmetry are compatible with the physical equivalence based on experimental errors. We can, however, easily present examples which break the third condition (transitivity), that is to say, $A \sim B$ and $B \sim C$ are satisfied within their errors but A and C does not coincide within their errors. This arises from the fact that real numbers which exceed any real numbers can be made from repeated additions of a non-zero real number because of Archimedian property of real number space.

"How can we introduce the physical equivalence in theories?"

Consistent definition of the physical equivalence is allowed, only when experimental errors are taken as *infinitesimal numbers*[1] in non-standard spaces. This result comes from the fact that any non-zero real numbers cannot be made from any finite sum of infinitesimal numbers. Any repetitions of the transitivity, that is, repeated additions of any infinitesimal numbers does not lead any non-zero real numbers. We can describe the situation as follows;

$$\forall \epsilon \in \mathrm{Mon}(0) \ \text{ and } \ \forall N \in \mathcal{N} \Longrightarrow \epsilon N \in \mathrm{Mon}(0),$$

where Mon(0) and $\mathcal{N}$, respectively, stand for the set of all infinitesimal numbers on non-standard spaces and the set of all natural numbers. From the above argument we can conclude that we must make theories, in which the physical equivalence based on experimental errors is described in terms of the mathematically consistent form, on a non-standard space. This is the reason why non-standard spaces are needed in the description of realistic theories based on the physical equivalence. It is once more stressed that such realistic theories must involve the fundamental observables, experimental errors, in the mathematically rigorous way.

An example for the introduction of the physical equivalence in quantum mechanics on non-standard space has been presented in the derivation of decoherence between quantum states for the description of quantum theory of measurements.[2-4] Though we have many other interesting problems for the construction of theories on non-standard spaces,[5-14] we shall investigate space-time structure and field theory in this paper.

Following the above argument, let us dicuss about observation of continuity of space-time. Whether space-time is continuous (as represented by the set of real numbers $\mathcal{R}$) or discrete (as represented by the set of discrete lattice-points) is a fundamental question for the space-time structure. We may ask

"How can we experimentally verify the continuous property of space-time?".

Taking into account that experimental errors are fundamental observables in physical phenomena, we should understand that the continuity of space-time cannot be directly verified in any experiments. This means that a discrete space-time is sufficient to describe realistic space-time. We, however, know that translational and rotational invariances (including Lorenz invariance) with respect to space-time axes seems to be very fundamental

concepts in nature and lattice spaces break them. This disadvantage seem to be very difficult to overcome on usual lattice spaces having a finite lattice-spacing between two neighboring lattice-points. As noted in the above argument, experimental errors must be described in terms of infinitesimal numbers on non-standard spaces. On non-standard spaces[1] we can introduce infinitesimal lengths which are smaller than all real numbers except 0. It will be an interesting question whether we can overcome the disadvantage on lattice spaces defined by infinitesimal lattice-spacing. Actually such infinitesimal discreteness cannot be observed in our experiments, where all results must be described by real numbers. This fact indicates that such lattice space-time will possibly be observed as continuous structure. Hereafter we call lattice spaces discretized by infinitesimal numbers *infinitesimal-lattice spaces* and they are denoted by ${}^*\mathcal{L}$.[11] That is to say, such a lattice space ${}^*\mathcal{L}$ is constructed as the set of non-standard numbers which are separated by an infinitesimal lattice-spacing ${}^*\varepsilon$ on ${}^*\mathcal{R}$(the non-standard extension of $\mathcal{R}$). It is transparent that such ${}^*\mathcal{L}$ do not contain many of real numbers in general. There is, however, a possibility that parts of infinitesimal neighborhoods of all real numbers are contained in ${}^*\mathcal{L}$, because it is known that the power of ${}^*\mathcal{L}$ is same as that of $\mathcal{R}$.[1] Thus there is a posibility that a space constructed from the set of all infinitesimal neighborhoods on ${}^*\mathcal{L}$ will be isomorphic to $\mathcal{R}$ and translations and rotations on the space can be introduced as same as those on $\mathcal{R}$.[11] We shall start from the investigation of properties of ${}^*\mathcal{L}$ and examine the construction of a new theory on the space-time represented by ${}^*\mathcal{L}$, where the space-time are not treated as parameters but written by operators. [12,13,14]

2 Infinitesimal-lattice spaces ${}^*\mathcal{L}$

Let us take a non-standard natural number ${}^*N \in {}^*\mathcal{N} - \mathcal{N}$, which is an infinity.[1] We take the closed set $[-{}^*N/2, {}^*N/2]$ on ${}^*\mathcal{R}$ and put $({}^*N)^2 + 1$ points with an equal spacing ${}^*\varepsilon = {}^*N^{-1}$ on the set. For the convenience of the following discussions ${}^*N$ is chosen as ${}^*N/2 \in {}^*\mathcal{N}$. The length between two neighboring points is ${}^*\varepsilon$ which is an infinitesimal, i.e. ${}^*\epsilon \in$Mon(0). Let us consider the set of the infinitesimal lattice-points ${}^*\mathcal{L}$,[11] which consists of these $({}^*N)^2 + 1$ discrete points on the closed set. Lattice-points on ${}^*\mathcal{L}$ are written by $l_n = n\,{}^*\varepsilon$, where $n \in {}^*\mathcal{Z}$ and fulfil the relation $-({}^*N)^2/2 \leq n \leq ({}^*N)^2/2$. From the process of the construction of ${}^*\mathcal{L}$ it is transparent that ${}^*\mathcal{L} \not\supset \mathcal{R}$. Actually it is obvious that all irrational numbers of $\mathcal{R}$ are not contained in ${}^*\mathcal{L}$, because ${}^*N$ is taken as an element of ${}^*\mathcal{N}$ and $n\,{}^*\epsilon = n/{}^*N$ is an element of ${}^*\mathcal{Z}$.

Let us show a theorem:

Monads of all real numbers, Mon*(r)* $\forall r \in \mathcal{R}$*, have their elements on* ${}^*\mathcal{L}$*.*

Proof: Take a real number $r \in \mathcal{R}$. The number r is contained in the closed set $[-{}^*N/2, {}^*N/2]$ on ${}^*\mathcal{R}$, because ${}^*N$ is an infinity of ${}^*\mathcal{N}$ and then $[-{}^*N/2, {}^*N/2] \supset \mathcal{R}$. Since the lattice-points of ${}^*\mathcal{L}$ divide the closed set into $({}^*N)^2$ regions of which lenght is ${}^*\varepsilon$, the real number r must be on a lattice-point or between two neighboring lattice-points whose distance is ${}^*\varepsilon$. We can, therefore, find out a non-standard integer N_r fulfilling the following relation;

$$N_r\,{}^*\varepsilon \leq r < (N_r + 1)\,{}^*\varepsilon, \tag{1}$$

where $|N_r| \in {}^*\mathcal{N} - \mathcal{N}$. The difference $r - N_r\ {}^*\varepsilon$ is an infinitesimal number smaller than ${}^*\varepsilon$. Thus we can define the infinitesimal neighborhood of r on ${}^*\mathcal{L}$ such that

$$\mathrm{Mon}(r|\ {}^*\mathcal{L}) \equiv \{l_n(r) = (N_r + n)\ {}^*\varepsilon | n \in {}^*\mathcal{Z},\ n\ {}^*\varepsilon \in \mathrm{Mon}(0)\}. \tag{2}$$

The relation of the standard part map[1] $\mathrm{st}(l_n(r)) = r$ is obvious. The theorem has been proved. Hereafter we shall call $\mathrm{Mon}(r|\ {}^*\mathcal{L})$ and its elements $l_n(r)$ monad lattice-space (${}^*\mathcal{L}$-monad) and monad lattice-points, respectively.

From the above argument we see that there is one-to-one correspondence between $\mathcal{R}$ and ${}^*\mathcal{L}_{l(\mathcal{R})} \equiv \{l_0(r)|r \in \mathcal{R}\}$ (the set of $l_0(r)$ for $\forall r \in \mathcal{R}$) with respect to the correspondence $r \leftrightarrow l_0(r)$. Note also that from the definition of monad we have the relations

$$\mathrm{Mon}(r|\ {}^*\mathcal{L}) \cap \mathrm{Mon}(r'|\ {}^*\mathcal{L}) = \phi, \quad \text{for}\ \ r \neq r',\ r, r' \in \mathcal{R}. \tag{3}$$

Magnitudes of lattice-points contained in all of the monad lattice-space $\mathrm{Mon}(r|\ {}^*\mathcal{L})$ for $\forall r \in \mathcal{R}$ are not infinity, because they are elements of monads of real numbers. We shall write the set of all these finite lattice-pionts by

$${}^*\mathcal{L}_\mathcal{R} \equiv \{l_n(r)|r \in \mathcal{R},\ n \in {}^*\mathcal{Z},\ n\ {}^*\varepsilon \in \mathrm{Mon}(0)\} = \cup_{r\in\mathcal{R}}\ \mathrm{Mon}(r|\ {}^*\mathcal{L}).$$

The sets ${}^*\mathcal{L}_\mathcal{R}$ and $\mathrm{Mon}(0|\ {}^*\mathcal{L})$ are additive groups. Note here that ${}^*\mathcal{L}_{l(\mathcal{R})}$ is not an additive group, because in general $l_0(r) + l_0(r') \neq l_0(r + r')$ possibly happens, that is, $N_{r+r'}$ is not always equal to $N_r + N_{r'}$ but possibly equal to $N_r + N_{r'} + 1$. It is apparent that ${}^*\mathcal{L}_\mathcal{R} = {}^*\mathcal{L}_{l(\mathcal{R})} + \mathrm{Mon}(0|\ {}^*\mathcal{L})$ and ${}^*\mathcal{L}_{l(\mathcal{R})} \cap \mathrm{Mon}(0|\ {}^*\mathcal{L}) = \{0\}$. Let us introduce the quotient set of ${}^*\mathcal{L}_\mathcal{R}$ by $\mathrm{Mon}(0|\ {}^*\mathcal{L})$ as

$${}^*\mathcal{M} \equiv {}^*\mathcal{L}_\mathcal{R}/\mathrm{Mon}(0|\ {}^*\mathcal{L}). \tag{4}$$

From one-to-one correspondence between $\mathcal{R}$ and ${}^*\mathcal{L}_{l(\mathcal{R})}$ we see that there is one-to-one correspondence between $\mathcal{R}$ and ${}^*\mathcal{M}$, and thus ${}^*\mathcal{M} \cong \mathcal{R}$ as additive groups, where the addition on ${}^*\mathcal{M}$ may be described by st-map of the addition on ${}^*\mathcal{L}_\mathcal{R}$ such that $\mathrm{st}(l_n(r) + l_m(r')) = r + r'$ for $\forall l_n(r) \in \mathrm{Mon}(r|\ {}^*\mathcal{L})$ and $\forall l_m(r') \in \mathrm{Mon}(r'|\ {}^*\mathcal{L})$ with $r, r' \in \mathcal{R}$.

We can introduce translations and rotations on ${}^*\mathcal{M}$ by using the relation ${}^*\mathcal{M} \cong \mathcal{R}$. (See refs. 11 and 14.)

3 Confined fractal-like property of ${}^*\mathcal{L}$

Though we have shown that ${}^*\mathcal{M} \cong \mathcal{R}$, there is a large difference between them, that is, ${}^*\mathcal{M}$ is constructed from the monad lattice-spaces $\mathrm{Mon}(r|\ {}^*\mathcal{L})$ which contain infinite number of different lattice-points on ${}^*\mathcal{L}_\mathcal{R}$. In fact the power of $\mathrm{Mon}(r|\ {}^*\mathcal{L})$ can be not countable but continuous in general. We can write the elements of $\mathrm{Mon}(r|\ {}^*\mathcal{L})$ as $l_n(r) = (N_r + n)\ {}^*\varepsilon$, where n can be elements of ${}^*\mathcal{N} - \mathcal{N}$, which satisfy the relation $n\ {}^*\varepsilon \in \mathrm{Mon}(0)$. There are a lot of different possibilities depending on the choice of the original non-standard natural number ${}^*N \in {}^*\mathcal{N} - \mathcal{N}$. We shall here show two examples, that is, one has an infinite series of ${}^*\mathcal{M}$ and the other a finite series.

(1) Infinite series of $^*\mathcal{M}$

Define an infinite series of infinite non-standard natural numbers by the following ultra-products;[1]

$$^*N_M \equiv \prod_{n\in\mathcal{N}} \alpha_n^{(M)}, \quad \text{for} \quad M \in \mathcal{N} \tag{5}$$

where $\alpha_n^{(M)} = 1$ for $0 \le n \le M$ and $\alpha_n^{(M)} = (n+1)^{n-M}$ for $n > M$. Following the definition of the order $>$ for ultra-products, we see that all of $^*N_M$ are infinity and the order is given by $^*N_0 > \ ^*N_1 > \ ^*N_2 > \cdots$. Then we have an infinite series of infinitesimal numbers $^*\varepsilon_0 < \ ^*\varepsilon_1 < \ ^*\varepsilon_2 < \cdots$, where $^*\varepsilon_M \equiv (^*N_M)^{-1}$. We can also prove that ratios

$$^*\lambda_M \equiv \frac{^*N_{M-1}}{^*N_M}, \quad \text{for} \quad M \ge 1 \tag{6}$$

are infinities of $^*\mathcal{N}$. Since $^*N_0$ is an element of natural numbers $^*\mathcal{N} - \mathcal{N}$, we can take $^*\varepsilon = \ ^*\varepsilon_0$. Here let us consider the following rescaling for the lattice points;

$$l_n(r) - l_0(r) = n \ ^*\varepsilon_0 \equiv \ ^*\lambda_1^{-1} l_n^{(1)}, \tag{7}$$

where $l_n^{(1)} = n \ ^*\varepsilon_1$. Note that $l_n^{(1)}$ is independent of r. Even if the relation $n \ ^*\varepsilon_0 \in \mathrm{Mon}(0)$ must be satisfied, the set of $n \in \ ^*\mathcal{N}$ satisfying the relation contains non-standard integers

$$n_m^{(1)} \equiv m \times \ ^*N_1 \in \ ^*\mathcal{Z}, \quad \text{for} \quad \forall m \in \mathcal{Z}. \tag{8}$$

It is trivial that the relation is satisfied as $n_m^{(1)} \ ^*\varepsilon_0 = m\lambda_1^{-1} \in \mathrm{Mon}(0)$. It is also obvious that $n_m^{(1)} \ ^*\varepsilon_1 = m \ \in \mathcal{Z}$. Thus we can see that the set of $\forall l_n^{(1)}$, $\ ^*\mathcal{L}_{\mathcal{R}}^{(1)} \equiv \{l_n^{(1)} = n \ ^*\varepsilon_1 | n \in \ ^*\mathcal{Z}, \ n \ ^*\varepsilon_0 \in \mathrm{Mon}(0)\}$, is an infinitesimal-lattice space with the lattice-length $^*\varepsilon_1$. In fact the set $^*\mathcal{L}_{\mathcal{R}}^{(1)}$ is constructed from the elements of $\mathrm{Mon}(r| \ ^*\mathcal{L})$ rescaled by the factor $^*\lambda_1$. From the facts that $^*\mathcal{L}_{\mathcal{R}}^{(1)}$ contains all integers, Archimedian property certifies the existence of natural numbers $m \ge |r|$ for $\forall r \in \mathcal{R}$ and the distance between two neighboring lattice-points is an infinitesimal number $^*\varepsilon_1$, we can find an element of $^*\mathcal{N} - \mathcal{N}$, $N_r^{(1)}$, satisfying the relation

$$N_r^{(1)} \ ^*\varepsilon_1 \le r^{(1)} < (N_r^{(1)} + 1) \ ^*\varepsilon_1, \quad \text{for} \quad \forall r^{(1)} \in \mathcal{R}. \tag{9}$$

Following the same argument for the construction of $^*\mathcal{M}$, we can introduce the monad of $r^{(1)}$, $\mathrm{Mon}(r^{(1)}| \ ^*\mathcal{L}_{\mathcal{R}}^{(1)})$, by the set of the following lattice-points on $^*\mathcal{L}_{\mathcal{R}}^{(1)}$;

$$l_n^{(1)}(r^{(1)}) = (N_r^{(1)} + n^{(1)}) \ ^*\varepsilon_1, \tag{10}$$

where $n^{(1)} \in \ ^*\mathcal{Z}$ and $\mathrm{st}(n^{(1)} \ ^*\varepsilon_1) = 0$ must be fulfilled. It is obvious that $\mathrm{Mon}(r^{(1)}| \ ^*\mathcal{L}_{\mathcal{R}}^{(1)})$ contains an infinite number of elements. Now we can define $^*\mathcal{M}^{(1)}$ by the set

$$^*\mathcal{M}^{(1)} \equiv \ ^*\mathcal{L}_{\mathcal{R}}^{(1)} / \mathrm{Mon}(0| \ ^*\mathcal{L}_{\mathcal{R}}^{(1)}). \tag{11}$$

The relation

$$^*\mathcal{M}^{(1)} \cong \ ^*\mathcal{M} \cong \mathcal{R} \tag{12}$$

as additive groups is obvious. Thus translations and rotations on N-dimensional space $({}^*\mathcal{M}^{(1)})^N$ are described as same as those of $({}^*\mathcal{M})^N$. We can conclude that every monad lattice-space $\mathrm{Mon}(r| {}^*\mathcal{L})$ for $\forall r \in \mathcal{R}$ contain the same space ${}^*\mathcal{M}^{(1)}$ by means of the same scale transformation.

By using the infinite series of ${}^*N_M$ we can proceed the same argument for the construction of ${}^*\mathcal{M}^{(M)}$ and thus we obtain the infinite series of sets isomorphic to $\mathcal{R}$ as additive group such that $\mathcal{R} \cong {}^*\mathcal{M} \cong {}^*\mathcal{M}^{(1)} \cong \cdots \cong {}^*\mathcal{M}^{(M)} \cong \cdots$.

(2) Finite series of ${}^*\mathcal{M}$

We definite a finite series of infinite numbers

$$ {}^*N_l^L \equiv \prod_{n\in\mathcal{N}} (n+1)^{L-l}, \quad \text{for } l = 0,1,2,\cdots,L-1 \tag{13} $$

where ${}^*N_l^L \in {}^*\mathcal{N} - \mathcal{N}$. We also see that

$$ {}^*\lambda_l \equiv \frac{{}^*N_{l-1}^L}{{}^*N_l^L} = \prod_{n\in\mathcal{N}} (n+1) \in {}^*\mathcal{N} - \mathcal{N}. \tag{14} $$

Following the same argument as that of the infinite series, we can construct a finite series of sets isomorphic to $\mathcal{R}$ as additive group $\mathcal{R} \cong {}^*\mathcal{M} \cong {}^*\mathcal{M}^{(1)} \cong \cdots \cong {}^*\mathcal{M}^{(L-1)}$.

4 Construction of fields on ${}^*\mathcal{M}$

Here we shall construct fields on ${}^*\mathcal{M}$. In the construction of field theory on ${}^*\mathcal{M}$ we follow the next two fundamental principles:[12,13,14]
(I) All definitions and evaluations should be carried out on the original space ${}^*\mathcal{L}$.
(II) In definitions of any kinds of physical quantities on ${}^*\mathcal{M}$, all the fields contained in the same monad lattice-space $\mathrm{Mon}(r|{}^*\mathcal{L})$ should be treated equivalently. (Principle of physical equivalece)
It should be noted that the principle (I) means that theories which we will make on ${}^*\mathcal{L}$ is generally not the same as any extensions of standard theories which have been constructed on $\mathcal{R}$. The principle (I) also tells us that all physical expectation values on $\mathcal{R}$ are obtained by taking standard part maps (maps from ${}^*\mathcal{R}$ to $\mathcal{R}$)[1] of results calculated on ${}^*\mathcal{L}$. The principle (II) is considered as the realization of the equivalence for indistinguishable quantities in quantum mechanics on non-standard space.[3] This principle, principle of physical equivalence, determines projections of physical systems defined on ${}^*\mathcal{L}$ to those defined on ${}^*\mathcal{M}$. Taking account of the fact that all points contained in the same monad lattice-space $\mathrm{Mon}(r|{}^*\mathcal{L})$ cannot be experimentally distinguished, the equivalent treatment with respect to all quantities defined on these indistinguishable points is a natural requirement in the construction of theories on ${}^*\mathcal{M}$.

(1) Fields on ${}^*\mathcal{L}$

Let us define two fields $A(m)$ and $\bar{A}(m)$ on every lattice point $r(m)$ on ${}^*\mathcal{L}$, which follow the commutation relations $[A(m), \bar{A}(m')] = \delta_{mm'}$ and others $= 0$. The vacuum

$|^*0>=\prod_m|0>_m$ and the dual vacuum $<\ ^*\bar{0}|=\prod_m\ {}_m<\bar{0}|$ are defined by $A(m)|0>_m=0$ and ${}_m<\bar{0}|\bar{A}(m)=0$ with ${}_m<\bar{0}|0>_m=1$. The fields $A(m)$ and $\bar{A}(m)$ operate only on the vacuum $|0>_m$ and the dual vacuum ${}_m<\bar{0}|$. Following the principle (I), all expectation values are imposed to be calculated on $^*\mathcal{L}$ such that $<\ ^*\bar{0}|\hat{\mathcal{O}}(\{A\},\{\bar{A}\})|^*0>\in\ ^*\mathcal{R}$, where $\hat{\mathcal{O}}$ is operator constructed from the sets of the fields $A(m)$ and $\bar{A}(m)$. Physical values are obtained by the standard part map[1] as $\mathrm{st}(<\ ^*\bar{0}|\hat{\mathcal{O}}|^*0>)\in\mathcal{R}$.

(2) Fields on $^*\mathcal{M}$

Following principle of physical equivalence (principle (II)), we define fields at every point on $^*\mathcal{M}$ as the following equivalent sum over all fields contained in $\mathrm{Mon}(r|^*\mathcal{L})$;

$$\varphi([r])\equiv\ ^*\sum_l A(N_r+l)/\sqrt{^*\sum_l 1},\qquad \bar{\varphi}([r])\equiv\ ^*\sum_l \bar{A}(N_r+l)/\sqrt{^*\sum_l 1},\tag{15}$$

where $^*\sum_l\ \equiv\ \sum_{l,^*\varepsilon l\in\mathrm{Mon}(0)}$ and hereafter $[r]$ in $\varphi([r])$ always means the fact that the equivalent sum over $\mathrm{Mon}(r|^*\mathcal{L})$ expressed by $^*\sum_l$ is carried out in the definition of $\varphi([r])$. Here the equivalent sum is just the expression of principle of physical equivalence. We can easily evaluate the commutation relation

$$[\varphi([r]),\bar{\varphi}([r'])]=\ ^*\delta_{rr'}=1\ (\text{for}\ \ r'=r),\quad =0\ (\text{for}\ \ r'\neq r).\tag{16}$$

Note that $r,r'\in\mathcal{R}$ but $^*\delta_{rr'}$ is not equal to the usual Dirac delta function $\delta(r-r')$. Complex fields on $^*\mathcal{M}$, which are represented by linear combinations certifying the same weight for all fields contained in $\mathrm{Mon}(r|^*\mathcal{L})$, are generally written by

$$\begin{aligned}\varphi([r];k)&=\ ^*\sum_l e^{i\theta_l^k(r)}A(N_r+l)/\sqrt{^*\sum_l 1},\\ \bar{\varphi}([r];k)&=\ ^*\sum_l e^{-i\theta_l^k(r)}\bar{A}(N_r+l)/\sqrt{^*\sum_l 1},\end{aligned}\tag{17}$$

where $\theta_l^k(r)=\theta_k(r)+2\pi lk/\ ^*\sum_l 1$ with the constraint $^*\varepsilon k\in\mathrm{Mon}(0)$ for non-standard integers k. They satisfy the commutation relations

$$[\varphi([r];k),\bar{\varphi}([r'];k')]=\ ^*\delta_{rr'}\delta_{kk'}\ \ \text{and}\ \ \text{others}=0.\tag{18}$$

These fields are the Fourier components for the fields on $\mathrm{Mon}(r|^*\mathcal{L})$ and their component number is same as that of $A(N_r+l)$ and $\bar{A}(N_r+l)$ included in $\mathrm{Mon}(r|^*\mathcal{L})$, because the constraint for k, that is, $^*\varepsilon k\in\mathrm{Mon}(0)$, is same as that for l. Note that $\varphi([r])$ and $\bar{\varphi}([r])$ correspond to the above fields with $k=0$ and $\theta_0(r)=0$. Experimentally the differences of the wave numbers k are not observable, because their wave lengths are infinitesimal. It is stressed that fields on one point of $^*\mathcal{M}$ have infinite degrees of freedom. General fields on $^*\mathcal{M}$ are described by functions of these fields $\phi([r])=f(\{\varphi([r];k)\},\{\bar{\varphi}([r];k)\})$.

(3) Extension to N-dimensional space

Extension of the fields to N-dimensional space $(^*\mathcal{M})^N$ is trivial. Note that one should not confuse the N for the N-dimensions of the space with the N_r for the lattice number

corresponding to r of $\mathcal{R}$ in the discussions. Every point of $({}^*\mathcal{L})^N$ is represented by a N-dimensional vector $\vec{r}^N(\vec{m}) \equiv (r_1(m_1), ..., r_N(m_N))$, where $r_i(m_i) = {}^*\varepsilon(N_{r_i} + l_i)$ with $\mathrm{st}({}^*\varepsilon N_{r_i}) = r_i \in \mathcal{R}$ and ${}^*\varepsilon l_i \in \mathrm{Mon}(0)$ for $i = 1, ..., N$. Fields with N-components at $\vec{r}^N(\vec{m})$, $A_j(\vec{m})$ and $\bar{A}_k(\vec{m})$ $(j, k = 1, ..., N)$, are defined by the commutation relations

$$[A_j(\vec{m}), \bar{A}_k(\vec{m}')] = \delta_{jk} \prod_{i=1}^{N} \delta_{m_i m_i'} \quad (\text{for} \quad j, k = 1, ..., N) \quad \text{and} \quad \text{others} = 0. \tag{19}$$

We may consider that these N number of fields describe N oscillators of a lattice point corresponding to N dimensional space-time axes. The fields on $({}^*\mathcal{M})^N$ are described as

$$\varphi_j([\vec{r}^N]; \vec{k}^N) = {}^*\sum_{l_1} \cdots {}^*\sum_{l_N} e^{i \sum_{s=1}^{N} \theta_{l(s)}^{k(s)}(\vec{r}^N)} A_j(N_{r_1} + l_1, \cdots, N_{r_N} + l_N) / ({}^*\sum_{l} 1)^{N/2} \tag{20}$$

and similar to $\bar{\varphi}_j([\vec{r}^N]; \vec{k})$. We again have the commutation relations

$$[\varphi_j([\vec{r}^N]; \vec{k}^N), \bar{\varphi}_l([\vec{r}'^N]; \vec{k}'^N)] = \delta_{jl} \prod_{i=1}^{N} ({}^*\delta_{r_i r_i'} \ \delta_{k_i k_i'}) \quad \text{and} \quad \text{others} = 0. \tag{21}$$

5 Internal symmetries on $({}^*\mathcal{M})^N$

Symmetries induced from the internal substructure $(\mathrm{Mon}(r|{}^*\mathcal{L}))^N$ on $({}^*\mathcal{M})^N$ are expressed by transformations U_T which keep all expectation values unchanged such that

$$< {}^*\bar{0}|\hat{\mathcal{O}}(\{A\}, \{\bar{A}\})|{}^*0 > = < {}^*\bar{0}|U_T^{-1} U_T \hat{\mathcal{O}}(\{A\}, \{\bar{A}\}) U_T^{-1} U_T|{}^*0 > .$$

In general the transformation U_T will be represented by maps of fields $A_j(\vec{m})$ $(\bar{A}_j(\vec{m}))$ to a linear combination of the fields $A_k(\vec{m})$ $(\bar{A}_k(\vec{m}))$ $(k = 1, \cdots, N)$ on ${}^*\mathcal{L}$. If the operators U_T do not change the structure of $({}^*\mathcal{M})^N$, they can represent symmetries on $(\mathrm{Mon}(r|{}^*\mathcal{L}))^N$.

(1) Transformation opertors on internal subspaces $(\mathrm{Mon}(r|{}^*\mathcal{L}))^N$

Let us start from the construction of transformation operators on an internal subspace contained in a point on $({}^*\mathcal{M})^N$ corresponding to a point $\vec{r}^N = (r_1, ..., r_N)$ on $\mathcal{R}^N$. The transformations map fields $A_j(\vec{r}^N(\vec{m}))$ $(\bar{A}_j(\vec{r}^N(\vec{m})))$ on every lattice-point $(\vec{r}^N(\vec{m}) = (N_{r_1} + l_1, ..., N_{r_N} + l_N))$ to linear combinations of fields $A_k(N_{r_1} + l_1', ..., N_{r_N} + l_N')$ $(\bar{A}_k(N_{r_1} + l_1', ..., N_{r_N} + l_N'))$ $(k = 1, ..., N)$ on the lattice-points of the same subspace. Following principle of physical equivalence (principle (II)), we construct the following N^2-number of operators $\hat{T}_{jk}([\vec{r}^N])$ on $({}^*\mathcal{M})^N$, which are again defined by the equivalent sum over all fields contained in the N-dimensional subspace $(\mathrm{Mon}(r|{}^*\mathcal{L}))^N$ as

$$\hat{T}_{jk}([\vec{r}^N]) = {}^*\sum_{l_1} \cdots {}^*\sum_{l_N} \bar{A}_j(N_{r_1} + l_1, ..., N_{r_N} + l_N) A_k(N_{r_1} + l_1, ..., N_{r_N} + l_N). \tag{22}$$

We easily obtain commutation relations

$$[\hat{T}_{jk}([\vec{r}^N]), A_l(\vec{r}'^N(\vec{m}))] = -(\prod_{i=1}^{N} {}^*\delta_{r_i r_i'}) \delta_{jl} A_k(\vec{r}^N(\vec{m})),$$

$$[\hat{T}_{jk}([\vec{r}^N]), \bar{A}_l(\vec{r}'^N(\vec{m}))] = (\prod_{i=1}^{N} {}^*\delta_{r_i r_i'})\delta_{kl}\bar{A}_j(\vec{r}^N(\vec{m})),$$

$$[\hat{T}_{jk}([\vec{r}^N]), \hat{T}_{lm}([\vec{r}'^N])] = (\prod_{i=1}^{N} {}^*\delta_{r_i r_i'})(\delta_{kl}\hat{T}_{jm}([\vec{r}^N]) - \delta_{jm}\hat{T}_{lk}([\vec{r}^N])). \tag{23}$$

These operators $\hat{T}_{jk}$ can be recomposed into the following generators;
(1) $U(1)$-generator: $\hat{J}_0 = \sum_{j=1}^{N} \hat{T}_{jj}$.
(2) $SU(N)$-generators: $\hat{J}_L = \sum_{j=1}^{L+1} g_j \hat{T}_{jj}$, for $L = 1, ..., N-1$ with the traceless condition $\sum_{j=1}^{L+1} g_j = 0$, and $\hat{J}^{(1)}_{jk} = \hat{T}_{jk} + \hat{T}_{kj}$ and $\hat{J}^{(2)}_{jk} = \frac{1}{i}(\hat{T}_{jk} - \hat{T}_{kj})$ for $j \neq k$. Now it is trivial that operators given by

$$U(\{\alpha(\vec{r}^N)\}) = \exp[i \sum_{j=1}^{N}\sum_{k=1}^{N} \alpha_{jk}(\vec{r}^N)\hat{T}_{jk}([\vec{r}^N])] \tag{24}$$

with st($\forall\alpha_{jk}(\vec{r}^N)$) $\in$ $\mathcal{C}$ (the set of complex numbers) produce maps of all fields on the subspace (Mon($r|$ ${}^*\mathcal{L}$))^N to linear combinations of the fields on the same subspace. From the construction procedure of $\hat{T}_{jk}$ it is obvious that the operators do not break the structure of (${}^*\mathcal{M})^N$. Note also that U does not change the vacuum and the dual vacuum, because $\forall\hat{T}_{jk}|^*0 >=<^* \bar{0}|\forall\hat{T}_{jk} = 0$.

(2) Symmetries on (${}^*\mathcal{M})^N$

Operators on (${}^*\mathcal{M})^N$ can be defined by products of $U(\{\alpha(\vec{r}^N)\})$ as

$$U_T(\{\alpha\}) = \prod_{i=1}^{N} {}^*\prod_{N_{ri}} U(\{\alpha(\vec{r}^N)\}), \tag{25}$$

where ${}^*\prod_{N_{ri}}$ stand for the product with respect to $\forall N_{r_i}$ with the constraint st(${}^*\varepsilon N_{r_i}$) $= r_i \in \mathcal{R}$. It is interesting that the transformations produced by $U_T(\{\alpha\})$ are generally local transformations on our observed space $({}^*\mathcal{M})^N$ because the parameters $\{\alpha\}$ can depend on the position $\vec{r}^N$, whereas they are global ones on the internal subspace (Mon($r|{}^*\mathcal{L}$))^N. Note that U_T does not change the vacuum and the dual vacuum.

Let us show a few realistic transformations included in U_T.
(a) $U(1)$ transformation:

$$U_0(\vec{r}^N) = \exp[i\alpha_0(\vec{r}^N)\hat{J}_0([\vec{r}^N])] \tag{26}$$

for st(α_0) $\in \mathcal{R}$. It is an interesting problem to investigate whether this $U(1)$ symmetry can be the $U(1)$ symmetry of electro-weak gauge theory or the solution of so-called $U(1)$ problem in hadron dynamics.
(b) $SU(N)$ transformation:

$$U_N(\vec{r}^N) = \exp[i\{\sum_{L=1}^{N-1} \alpha_L(\vec{r}^N)\hat{J}_L([\vec{r}^N]) + \sum_{j=1}^{k-1}\sum_{k=2}^{N}\sum_{i=1}^{2} \alpha^{(i)}_{jk}(\vec{r}^N)\hat{J}^{(i)}_{jk}([\vec{r}^N])\}] \tag{27}$$

for st($\forall\alpha_L$), st($\forall\alpha^{(i)}_{jk}$) $\in \mathcal{R}$. It is an interesting proposal that three color components of QCD may be identified by those of $U_3(\vec{r}^3)$ for three spatial dimensions.

6 Quantized configuration space

In usual field theory space-time variables are treated as parameters. Here we construct configuration space describing ${}^*\mathcal{M}$, where the space-time are expressed by operators.

(1) Quantization of configuration space

We can construct position operator for 1-dimensional space

$$\hat{r}_{\ {}^*\mathcal{M}} = {}^*\sum_{N_r} r\hat{T}_r, \tag{28}$$

where ${}^*\sum_{N_r}$ stands for the sum over $\forall N_r$ with the constraint $\mathrm{st}({}^*\varepsilon N_r) = r \in \mathcal{R}$ and $\hat{T}_r = {}^*\sum_l \bar{A}(N_r+l)A(N_r+l)$. Following principle of physical equivqlence, $\hat{T}_r$ is expressed by the equivalent sum with respect to all fields in the same monad lattice-space $\mathrm{Mon}(r|\ {}^*\mathcal{L})$. Note that r in (77) can be replaced by $r + a_r\ {}^*\varepsilon$ with the constant $\mathrm{st}(a_r\ {}^*\varepsilon) = 0$ for $\forall r \in \mathcal{R}$. The eigenstate of $\hat{r}_{\ {}^*\mathcal{M}}$ for the eigenvalue r is written by

$$|r>_{\ {}^*\mathcal{M}} \equiv \bar{\varphi}([r])|{}^*0> . \tag{29}$$

Hereafter we call them monad states. The relation $\hat{r}_{\ {}^*\mathcal{M}}|r>_{\ {}^*\mathcal{M}} = r|r>_{\ {}^*\mathcal{M}}$ is trivial. If one does not want to have 0 eigenvalue for $r = 0$, $r + a_r\ {}^*\varepsilon$ can be used instead of r in the definitoin of $\hat{r}_{\ {}^*\mathcal{M}}$. The monad states $|r>_{\ {}^*\mathcal{M}}$ are quite similar to the ket states of usual quantum mechanics except the normalization condition ${}_{\ {}^*\mathcal{M}}<r|r'>_{\ {}^*\mathcal{M}} = {}^*\delta_{rr'}$, where ${}_{\ {}^*\mathcal{M}}<r| = <{}^*\bar{0}|\ {}^*\prod_{N_r}\varphi([r])$. It is noted that every monad state $|r>_{\ {}^*\mathcal{M}}$ has its own internal substructure $\mathrm{Mon}(r|\ {}^*\mathcal{L})$.

Now we can define the quantized states for our configuration space as follows;

$$|\ {}^*\mathcal{M}> \equiv {}^*\prod_{N_r}|r>_{\ {}^*\mathcal{M}}, \quad <{}^*\mathcal{M}| \equiv {}^*\prod_{N_r}{}_{\ {}^*\mathcal{M}}<r|. \tag{30}$$

On these states the position operator $\hat{r}_{\ {}^*\mathcal{M}}$ is represented by a diagonal operator and then we can consider that the base state $|\ {}^*\mathcal{M}>$ describes our configuration space, which is normalized as $<{}^*\mathcal{M}|\ {}^*\mathcal{M}> = 1$.

Extension to N-dimension is trivial. A component of the position-vector operator can be defined as same as that of the 1-dimensinal case, e.g., for the ith component

$$\hat{r}_{i\ {}^*\mathcal{M}} = {}^*\sum_{N_{r1}}\cdots {}^*\sum_{N_{rN}} r_i\hat{T}_i([\vec{r}^N]), \tag{31}$$

where $\hat{T}_i([\vec{r}^N]) = {}^*\sum_{l_1}\cdots {}^*\sum_{l_N}\bar{A}_i(N_{r_1}+l_1,...,N_{r_N}+l_N)A_i(N_{r_1}+l_1,...,N_{r_N}+l_N)$ for $i = 1,2,...,N$. The N-dimensional configuration state is expressed by $|\ {}^*\mathcal{M}^N> = \prod_{j=1}^N({}^*\prod_{N_{r1}}\cdots {}^*\prod_{N_{rN}}\bar{\varphi}_j([\vec{r}^N]))|{}^*0>$.

(2) Infinitesimal distance

We can define infinitesimal relative distance operators only on the internal subspace $\mathrm{Mon}(r|{}^*\mathcal{L})$ such that

$$d\hat{r}(\Delta l) \equiv \hat{r}(N_r+l) - \hat{r}(N_r+l'), \tag{32}$$

where $\Delta l \equiv l - l'$ and $\hat{r}(N_r + k) \equiv {}^*\varepsilon(N_r + l)\bar{A}([r])A(N_r + k)$ with the definiion $\bar{A}([r]) \equiv {}^*\sum_l \bar{A}(N_r + l)$, which follows princilpe of physical equivalnce. The monad states $|r >_{^*\mathcal{M}}$ are the eigenstates of $\hat{r}(N_r + l)$ and $d\hat{r}(\Delta l)$. We actually obtain

$$d\hat{r}(\Delta l)|r >_{^*\mathcal{M}} = {}^*\varepsilon \Delta l |r >_{^*\mathcal{M}} . \tag{33}$$

We can write squared distance operators in the N-dimensional space as $(d\hat{s})^2(\vec{r}^N) = d\hat{r}_\mu(\Delta\vec{l}^N)g^{\mu\nu}d\hat{r}_\nu(\Delta\vec{l}^N)$, where the sums over μ and ν from 1 to N are neglected,

$$d\hat{r}_\mu(\Delta\vec{l}^N) = \hat{r}_\mu(N_{r_1} + l_1, ..., N_{r_N} + l_N) - \hat{r}_\mu(N_{r_1} + l'_1, ..., N_{r_N} + l'_N)$$

with $\hat{r}_\mu(N_{r_1} + l_1, ..., N_{r_N} + l_N) = {}^*\varepsilon(N_{r_\mu} + l_\mu)\bar{A}_\mu([\vec{r}^N])A_\mu(N_{r_1} + l_1, ..., N_{r_N} + l_N)$ and $\Delta\vec{l}^N = (l_1 - l'_1, \cdots, l_N - l'_N)$. If the metric oprator $g^{\mu\nu}$ is taken as Minkowski metric, the internal subspace $(\mathrm{Mon}(r|{}^*\mathcal{L}))^N$ just represents so-called local inertial system in general relativity. We have the equations

$$\begin{aligned} d\hat{r}_\mu(\Delta\vec{l}^N)|\vec{r}^N >_{^*\mathcal{M}} &= {}^*\varepsilon \Delta l_\mu |\vec{r}^N >_{^*\mathcal{M}}, \\ (d\vec{s})^2(\vec{r}^N)|\vec{r}^N > &= {}^*\varepsilon^2 \Delta l_\mu g^{\mu\nu} \Delta l_\nu |\vec{r}^N >_{^*\mathcal{M}} . \end{aligned} \tag{34}$$

The expectation value of $(d\hat{s})^2$ is calculated as $(ds)^2 = {}_{^*\mathcal{M}}< \vec{r}^N|(d\hat{s})^2(\vec{r}^N)|\vec{r}^N >_{^*\mathcal{M}}$. The same expectaton value of the squared distance operator can be obtained in terms of the expectation value with respect to the configuration state $|\,{}^*\mathcal{M}^N >$. It is transparent that transformations keeping $(ds)^2$ unchanged are represented by $U(\{\alpha(\vec{r}^N)\})$.

7 Translations, Rotations and relativistic transformations

We shall study symmetries on the configuration space, which keep all expectation values unchanged such that $<\,{}^*\mathcal{M}^N|U^{-1}U\hat{\mathcal{O}}(\{\bar{A}\}, \{A\})U^{-1}U|\,{}^*\mathcal{M}^N >$. Note that the configuration state $|\,{}^*\mathcal{M}^N >$, the dual state $<\,{}^*\mathcal{M}^N|$ and operators are transformed as

$$|\,{}^*\mathcal{M}^N > \longrightarrow U|\,{}^*\mathcal{M}^N >, \quad <\,{}^*\mathcal{M}^N| \longrightarrow <\,{}^*\mathcal{M}^N|U^{-1}, \quad U\hat{\mathcal{O}}(....)U^{-1}.$$

(1) Translational invariance on $(\,{}^*\mathcal{M})^N$

The operator which replaces $|r>$ with $|r + \Delta >$ for $\Delta \in \mathcal{R}$ is obtained as

$$\hat{p}_r(\Delta) = {}^*\sum_l \bar{A}(N_{r+\Delta} + l)A(N_r + l). \tag{35}$$

We have $\hat{p}_r(\Delta)|{}^*0 >= 0$. Then we can define the translation operator by

$$\hat{P}(\Delta) =: {}^*\prod_{N_r} \hat{p}_r(\Delta) :, \tag{36}$$

where : ...... : means the normal product used in usual field theory, in which all creation operators $(\bar{A}_j(m))$ must put on the left-hand side of all annihilation operators $(A_j(m))$.

We see that $\hat{P}(\Delta)$ transforms the configuration state $|\ ^*\mathcal{M}>$ to the isomorophic space $\hat{P}(\Delta)|\ ^*\mathcal{M}> \cong |\ ^*\mathcal{M}>$ for st$(\forall\Delta) \in \mathcal{R}$.

Let us study the invariance of expectation values $<\ ^*\mathcal{M}|\hat{\mathcal{O}}(\{\bar{A}\}, \{A\})|\ ^*\mathcal{M}>$. Taking account of the definitions of $|\ ^*\mathcal{M}>= \prod \bar{\varphi}([r])|\ ^*0>$ and $<\ ^*\mathcal{M}| =<\ ^*\bar{0}|\prod \varphi([r])$ and the fact that all the fields commute each other except A and $\bar{A}$ on the same lattice-point, the number of A and that of $\bar{A}$ on the same lattice-point must be same in operators having non-vanishing expectation values on $|\ ^*\mathcal{M}>$. This means that every term of such operators must be written by the product of powers such as $(\bar{A}A)^n$ with $n \in \mathcal{N}$ for all pairs of A and $\bar{A}$ on the same lattice-point. On the other hand we easily see that the products of $\bar{A}A$ on the same lattice-piont commute with $\hat{P}(\Delta)$ such that $[\bar{A}A, \hat{P}(\Delta)] = 0$ for $\forall\Delta \in \mathcal{R}$. Now we can conclude that operators having non-vanishing expetation values commute with the translation operators, that is, $[\hat{\mathcal{O}}(\{\bar{A}\}, \{A\}), \hat{P}(\Delta)] = 0$. Translational invariance is certified for physically meaningful operators as

$$<\ ^*\mathcal{M}|\hat{P}(-\Delta)\hat{\mathcal{O}}(...)\hat{P}(\Delta)|\ ^*\mathcal{M}>=<\ ^*\mathcal{M}|\hat{\mathcal{O}}(...)|\ ^*\mathcal{M}>, \tag{37}$$

where the commutatibity of $\hat{\mathcal{O}}$ and $\hat{P}$ and $<\ ^*\mathcal{M}|\hat{P}(-\Delta)\hat{P}(\Delta) =<\ ^*\mathcal{M}|$ are used.

The extension of the above argument to the N-dimensional spaces is trivial.

(2) Rotations

Rotational invariance can be introduced only for subspaces whose metric $g^{\mu\nu}$ have the same sign like $SO(3)$ subspace of $SO(3,1)$. Generators for the rotations in (j,k)-plane are given by $\hat{J}_{jk} = \hat{T}_{jk} - \hat{T}_{kj}$. In general rotation operators are described by

$$U_R(\{\theta\}) = e^{i\sum_{(j,k)}\theta_{jk}\hat{J}_{jk}}. \tag{38}$$

We see that U_R for st$(\forall\theta_{jk}) \in \mathcal{R}$ are unitary and generate rotations on the subspace.

(3) Lorentz transfomations

Position operator for one point on $(\ ^*\mathcal{M})^N$ corresponding to $\vec{r}^N$ on $\mathcal{R}^N$ is given by

$$\hat{r}_j([\vec{r}^N]) = r_j\bar{\varphi}_j([\vec{r}^N])\varphi_j([\vec{r}^N]), \quad \text{for } j = 1, ..., N. \tag{39}$$

The expectation value of squared distance from the origin are evaluated as $(\vec{r}^N)^2 =<\ ^*\mathcal{M}^N|\hat{r}_\mu([\vec{r}^N])g^{\mu\nu}\hat{r}_\nu([\vec{r}^N])|\ ^*\mathcal{M}^N>$, where the metric tensors $g^{\mu\nu}$ are taken as Minkowski metric tensors. Let us study the simplest case for $N = 2$. The metric tensors are chosen such that $g^{11} = -g^{22} = 1$ and $g^{12} = g^{21} = 0$. Transformations

$$U_L(a) = \prod_{j=1}^{N} {}^*\prod_{N_{rj}} e^{-a\hat{J}_{12}^{(1)}([\vec{r}^N])} \tag{40}$$

with the constraint st$(a) \in \mathcal{R}$ generate 2-dimensional Lorentz transformations which are expressed in 2-dimensional matrices as

$$U_L(a) = \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix}$$

Generalization for the N-dimensions can be performed by using combinations of $U_L(a)$ with the rotations.

(4) General relativistic transformations

We have many different types of transformations which keep the squred distance $(\vec{r}^N)^2$ invariant but generally do not the metric tensors invariant, while Lorentz transformations keep both of them invariant. They are described by the transformations $U_T(\{\alpha\})$, where the parameters $\{\alpha\}$ should be chosen such that all the axes are real after the translations. Of course, all the parameters must be finite. In such transformations we have different types of vectors corresponding to covariant and contravariant tensors in general coordinate transformations. The difference between them is expressed as follows;

$$\begin{aligned} U_G \hat{r}_\mu | \,^*\mathcal{M}^N >, \qquad & \text{for covariant vectors} \\ < \,^*\mathcal{M}^N | \hat{r}_\mu g^{\mu\nu} U_G^{-1}, \qquad & \text{for contravariant vectors.} \end{aligned} \tag{41}$$

A simple example representing dilatation transformations are described by

$$D_d = e^{\sum_{j=1}^N a_j(\vec{r}^N) \hat{T}_{jj}([\vec{r}N])} \tag{42}$$

with $\forall a_j(\vec{r}^N) \in \,^*\mathcal{R}$, which transforms as

$$U_d \ \hat{r}_\mu | \,^*\mathcal{M}^N >= e^{a_\mu(\vec{r}^N)} \hat{r}_\mu | \,^*\mathcal{M}^N >,$$

$$< \,^*\mathcal{M}^N | \hat{r}_\nu g^{\nu\mu} U_d^{-1} =< \,^*\mathcal{M}^N | \hat{r}_\nu g^{\nu\mu} e^{-a_\mu(\vec{r}^N)}.$$

We see that these transformations change the eigenvalues of the covariant and the contravariant vectors.

Note that $U_G(\{\alpha(\vec{r}^N)\})$ is global on the subspace $(\mathrm{Mon}(r| \,^*\mathcal{L}))^N$, while it is generally local on observed space $(\,^*\mathcal{M})^N$. Note also that all the transformations described by U_T can include general relativistic transformations. This fact implies that general relativstivc transformations are generally represented by local non-abelian transformations.

8 Concluding remarks

We shall briefly comment that, instead of bosonic fields $A(m)$ and $\bar{A}(m)$, we can construct similar field theory by using fermionic fields $C(m)$ and $\bar{C}(m)$ which satisfy anti-commutaion relations $[C(m), \bar{C}(m)]_+ = 1$ and commutation relations $[C(m), C(m')]_- = [C(m), \bar{C}(m')]_- = [\bar{C}(m), \bar{C}(m')]_- = 0$ for $m \neq m'$. As far as operators $\hat{T}_{jk}([\vec{r}^N])$ are concerned, we can define them by the replacement of A and $\bar{A}$ with C and $\bar{C}$, respectively. And we get the same commutation relations. This means that all the arguments of the internal symmetries performed in the bosonic oscillator case are completely accomplished in the fermionic oscillator case. That is to say, as far as the internal symmetries are concerned, there is no difference between the bosonic and the fermionic cases. Futhermore we can easily understand that not only U_T but also all other operators written by the products of $\bar{A}$ and A like $\hat{T}_r$, $\hat{r}$ and $\hat{p}_r$ can be defined in the replacement of $\bar{A}$ A

with $\bar{C}\ C$ and they have the same properties as discussed in the bosonic case. Difference between them appears in the construction of realistic fields from $\varphi([r];k)$. Namely products of more than the non-standard natural number ${}^*\sum_l 1$ with respect to the fields $\varphi([r];k)$ vanish for the fermionic case, whereas there is no such restriction in the bosonic case. We may say that the concept of antiparticles will be introduced more easily in the fermionic case by using occupaton and unoccupation numbers of lattice-points of the monad lattice-space $\mathrm{Mon}(r|\ {}^*\mathcal{L})$. Anyhow the selection of the bosonic or the fermionic or both like supersymmetric is still open question at present.

We have constructed a field thoery on the quantized space-time by using infinitesimal-lattice space $(\ {}^*\mathcal{L})^N$. In this scheme the internal subspace $(\mathrm{Mon}(r|{}^*\mathcal{L}))^N$ and the symmety transformation U_T induced from the subspace are uniquely determined, when we construct the field theory on $({}^*\mathcal{M})^N \cong \mathcal{R}^N$. Since all definitions and evaluations are imposed to be done on $(\ {}^*\mathcal{L})^N$, we can perform them in terms of $*$-finte sum in non-standard analysis. In fact we need not introduce any Dirac δ-functions. In this scheme we can carry out all evaluations on configuration space, not on Fock space in usual field theory. This fact is an interesting advantage in the investigation of quantum gravity, as was seen in the introduction of the infinitesimal relative distance and the local inertial system. In order to investigate this model in more detail an inevitable problem is introducing equation of motions on $({}^*\mathcal{M})^N$, which will be represented by difference equation on $\mathrm{Mon}(r|{}^*\mathcal{L})$. It is also interesting to study relations between the general field $\phi([r])$ and observed fields like leptons, quarks, gauge fields and etc.

Finally I would like to present the global view of theory on non-standard space once more. The fundamental concept is introducing the equivalence based on experimental errors (physical equivalence) into theories in a mathematically consistent logic, which is allowed only on non-standard spaces. On the spaces the physical equivalence determine projections from non-standard spaces to observed spaces isomorphic to $\mathcal{R}^N$, which are described by filters in non-standard theory. In fact the filters determine topologies, because they determine the structure of the monad space and then that of the observed space.[1] We have to understand that in an experiment we are allowed to peep only through a filter which is determined by the physical equivalence based on the errors of the experiment. Theories on observed spaces, which explain experimental results, of course have to depend on the filters which determine the projections of the theory on the non-standard space to theories on the observed spaces, even if the theory is uniqe on the non-standard space. Different filters derive different monad spaces and then different observed spaces (different theories). I would like again to repeat that we cannot perform any expriments which are not accompanied by any errors. Therefore we have always to take account of phenomena hidden behind experimental errors, when we make theories in our observed spaces.

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